Dimension

Recall that the dimension of a ring R is the supremum of lengths of chains of prime ideals. If $I \subseteq R$ is an ideal, we say the dimension of I (or of $V(I)$) is $\dim(R/I)$.

One of our goals will be to show $\dim(k[x_1,\dots,x_n]) = n$. (Recall that we can bound it below by n.)

Def: If $P \subseteq R$ is prime, the codimension or height of P is the supremum of lengths of chains of primes descending from P. (Note that $\operatorname{codim} P = \dim R_P$.)

For $I \subsetneq R$ any proper ideal, define

$$\operatorname{codim} I := \min\{\operatorname{codim} P \mid P \supseteq I \text{ is prime}\}$$

We can extend the notion of dimension to modules:

Def: Let M be an R-module. The dimension of M is

$$\dim M := \dim \operatorname{Ann} M = \dim R/\operatorname{Ann} M.$$

Caution: This can be confusing. If $I \subseteq R$ an ideal in an integral domain, then if we consider I as an R-module, we have $\operatorname{Ann} I = 0$, so

$$\underset{\substack{\uparrow \\ \text{as an} \\ R\text{-module}}}{\dim I} = \dim \operatorname{Ann} I = \dim R/0 = \dim R,$$

whereas $\underset{\substack{\nwarrow \\ \text{as an ideal}}}{\dim I} = \dim R/I.$

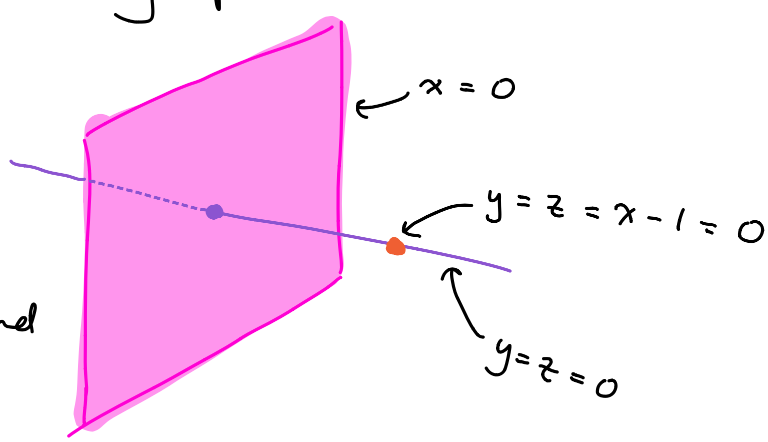
So when we write $\dim I$, we typically mean the dim as an ideal.

Question: Let $I \subseteq R$ be an ideal. Why can't we define $\operatorname{codim} I = \dim R - \dim I$? (Note that $\dim I + \operatorname{codim} I \leq \dim R$)

Answer: We'll see that this is true in "nice" cases (e.g. if R is a domain, f.g. as a k-algebra), but it's not true more generally:

EX: Consider the following picture in $\mathbb{A}^3$:

In $k[x,y,z]$, the union of the yz-plane and the x-axis is cut out by the ideal $(x)(y,z) = (xy, xz)$.



Let $R = k[x,y,z]/(xy, xz)$

Then $(x) \subsetneq (x,y) \subsetneq (x,y,z)$ are all prime, so $\dim R \geq 2$.
(In fact, $\dim R = 2$).

However, if we set $I = (x-1, y, z) \subseteq R$, then

$$\dim I = \dim R/I = 0, \text{ since } I \text{ is max'l.}$$

But we have

$$\operatorname{codim} I = \dim R_I = \dim k[x,y,z]_I/(y,z) \cong \dim k[x]_{(x-1)} = 1.$$

so $\dim I + \operatorname{codim} I \neq \dim R$.

Intuition: $\dim R$ is the dimension of the largest component, whereas $\operatorname{codim} I$ gives the "local" codimension. i.e. the codim of I in the component it lies in.

Connection to Artinian rings

Recall that we showed that

R is Artinian $\iff$ R is Noetherian and all prime ideals are maximal.

Moreover, R will have only finitely many prime ideals.

Since $\dim R = 0 \Leftrightarrow$ all prime ideals are max'l, we can translate this as follows:

Cor: R Artinian $\Leftrightarrow$ R Noetherian and $\dim R = 0$. Moreover, if R is Artinian, $\operatorname{Spec} R$ consists of finitely many points, all closed.

The principal Ideal Theorem

For the rest of the section, assume all rings are Noetherian.

Question: If $I = (a_1, \dots, a_n)$, what is $\operatorname{codim} I$? Even more basic: What's $\operatorname{codim} (a)$?

We can easily find the codim of primes contained in principal ideals:

Claim: Any prime P properly contained in a principal ideal $(x) \subsetneq R$ has codim 0.

Pf: Suppose $Q \subsetneq P \subsetneq (x)$ with Q prime. Then R/Q is an integral domain. So replace R with R/Q and assume R is an int. domain.

If $y \in P$, then $y = ax$ for some a. $x \notin P$, so $a \in P$. Thus $P = xP$. We can't apply Nakayama (x may not be in $J(R)$), but we showed via C-H that

$$(1-b)P = 0 \text{ for some } b \in (x).$$

R is an int. domain, so $b = 1$, a contradiction. $\square$

Krull's principal ideal theorem extends this to primes minimal over principal ideals:

Krull's principal ideal Theorem: If $x \in R$ and P is minimal among primes containing x, then $\operatorname{codim} P \leq 1$.

For the proof of the PIT, we need one more tool: symbolic powers of ideals.

Def: Let $Q \subseteq R$ be prime. The n^{th} symbolic power of Q is

$$Q^{(n)} := \underbrace{\underbrace{Q^n R_Q}_{\text{expansion}} \cap R}_{\text{contraction}} = \{ r \in R \mid sr \in Q^n \text{ for some } s \in R - Q \}$$

Recall that $Q^n \subseteq Q^{(n)} \subseteq Q$ and $(Q^{(n)})_Q = (Q_Q)^n$ (exer)

sometimes equality, but not always (recall previous HW problem)

Ex: Let $R = k[x,y,z]/(xy-z^2)$ and $P = (x,z) \subseteq R$ prime.

$$P^2 = (x^2, xz, \underset{= xy}{z^2}), \quad \text{so} \quad P^2 R_P = (x^2, xz, x\underset{\text{unit}}{y}) = (x).$$

Thus, $P^{(2)} = \left(\frac{x}{1}\right) \cap R = (x^2, xz, \underset{xy}{\overset{\shortparallel}{z^2}}, x) = (x)$

So $P^2 \underset{\substack{\uparrow \\ \text{contains} \\ x}}{\subsetneq} P^{(2)} \underset{\substack{\uparrow \\ \text{contains} \\ z}}{\subsetneq} P$

Proof of the PIT: Let $x \in R$, P minimal among primes containing x. We'll show that if $Q \subsetneq P$ is prime, then $\dim R_Q = 0$, so $\operatorname{codim} Q = 0$. This shows $\operatorname{codim} P \leq 1$.

Since $\operatorname{codim} P = \dim R_P = \operatorname{codim} PR_P$, we can replace R with R_P and assume R is local and P maximal.

Since P is minimal over (x), we have, by a previous corollary about Artinian rings, that $R/(x)$ is Artinian.

Thus, the chain

$$(x) + Q \supseteq (x) + Q^{(2)} \supseteq (x) + Q^{(3)} \supseteq \cdots$$

eventually stabilizes. Say $(x) + Q^{(n)} = (x) + Q^{(n+1)}$.

Then, in particular, $Q^{(n)} \subseteq (x) + Q^{(n+1)}$, so for $f \in Q^{(n)}$, We can write $f = ax + g$, with $g \in Q^{(n+1)}$

$$\Rightarrow \quad ax = f - g \in Q^{(n)} = Q^n R_Q \cap R$$

$$\Rightarrow \quad \frac{ax}{1} \in Q^n R_Q$$

$$\Rightarrow \quad bax \in Q^n \quad \text{for some } b \notin Q.$$

$x \notin Q$ by minimality of P, so $bx \notin Q$.

Thus, $\frac{a}{1} \in Q^n R_Q \implies a \in Q^{(n)}$.

So $$f = \underset{\text{in } Q^{(n)}}{a}x + \underset{\text{in } Q^{(n+1)}}{g} \in (x)Q^{(n)} + Q^{(n+1)}$$

$$\Rightarrow \quad Q^{(n)} \subseteq \underbrace{(x)Q^{(n)}} + \underbrace{Q^{(n+1)}}.$$

Both of these ideals are in $Q^{(n)}$, so the reverse inclusion holds, and we have

$$Q^{(n)} = (x)Q^{(n)} + Q^{(n+1)}.$$

Thus, if $\overline{Q^{(n)}}$ is the image of $Q^{(n)}$ in $R/Q^{(n+1)}$, we have $\overline{Q^{(n)}} = (x)\overline{Q^{(n)}}$.

Since R is local, $(x) \subseteq J(R)$. Since R is Noetherian, $Q^{(n)}$, and thus $\overline{Q^{(n)}}$, is finitely generated. So we can apply Nakayama, and get

$$\overline{Q^{(n)}} = 0, \text{ so } Q^{(n)} = Q^{(n+1)}.$$

Thus, $\underset{\substack{\shortparallel \\ Q_Q^n}}{Q^n R_Q} = \underset{\substack{\shortparallel \\ Q_Q^{n+1}}}{Q^{n+1} R_Q}$, which again by Nakayama,

says that $Q_Q^n = 0$.

The corollary about Artinian rings says that $R_Q/0 = R_Q$ must thus be Artinian, so $\operatorname{codim} Q = \dim R_Q = 0$. $\square$

We can use this as the base case for an induction involving primes minimal over a f.g. ideal:

Theorem: If $I = (x_1, \ldots, x_c) \subseteq R$, and P is minimal among primes containing I, then $\operatorname{codim} P \leq c$.

Pf: Again, $\operatorname{codim} P = \dim R_P$, so assume R is local w/ max'l ideal P.

Then $P^n \subseteq I$ for $n >> 0$ (again by cor. about Artinian rings).

Let P_1 be a prime s.t. $P_1 \subseteq P$ with no primes in between.

We will show that P_1 is minimal over an ideal generated by $c-1$ elements. By induction, $\operatorname{codim} \leq c-1$, and we're done.

By minimality of P, P_1 cannot contain all the x_i. WLOG, assume $x_1 \notin P_1$. Then P is minimal over (P_1, x_1), so $P^n \subseteq (P_1, x_1)$ for $n >> 0$, so in particular, $x_i^n \in (P_1, x_1)$ for all i.

That is, for each i, we can find $a_i \in R$, $y_i \in P_1$ s.t.

$$x_i^n = a_i x_1 + y_i.$$

Let $J = (y_2, \dots, y_c)$. Then for some $m >> 0$, $I^m \subseteq (x_1, J)$

So $P^{n+m} \subseteq I^m \subseteq (x_1, J)$, so P is minimal over (x_1, J).

Thus, the image of P in R/J is minimal over $(\bar{x}_1)$.

So the PIT says $\operatorname{codim} \bar{P} = \dim (R/J)_P = \dim R/J \leq 1$.

So since $J \subseteq P_1 \subsetneq P$, P_1 must be minimal over J, which is generated by $c-1$ elements, so we're done. $\square$

This immediately gives us a descending chain condition for prime ideals in a Noetherian ring:

Cor: Let P be a prime ideal in a Noetherian ring. Then any strictly decreasing chain of prime ideals

$$P \supsetneq P_1 \supsetneq P_2 \supsetneq \cdots$$

has finite length, bounded above by the number of generators of P.

In particular, since $(x_1, \dots, x_c) \subseteq k[x_1, \dots, x_n]$ has descending chain $(x_1, \dots, x_c) \supsetneq (x_2, \dots, x_c) \supsetneq \cdots \supsetneq (x_c) \supsetneq 0$, we know its codim is $\geq c$. The above gives an upper bound, so we have:

Cor: The ideal $(x_1, \dots, x_c) \subseteq k[x_1, \dots, x_n]$ has codimension c.

This doesn't quite suffice yet to compute the dim of the poly. ring, but we'll be able to soon.

There's a partial converse to the PIT:

Cor: If P is prime of codim c, then P is minimal over an ideal generated by c elements.

Pf: By induction on c. If $c = 0$, any prime of codim 0 is minimal over 0, so assume $c \geq 1$.

Let $Q_1, \dots, Q_n$ be the minimal primes contained in P. Then they are minimal over $0 = \mathrm{Ann}(R/0)$, so they are associated primes of 0. Since R is Noetherian, there must only be finitely many.

$P \neq Q_i$ for any i since $c \geq 1$. Thus, by Prime avoidance

$$P \neq \bigcup Q_i .$$

So we can find $x_1 \in P \setminus (\bigcup Q_i)$.

Primes in $P/(x_1)$ correspond to primes in P containing x_1, so $\operatorname{codim} P/(x_1) \leq c-1$. By induction, $P/(x_1)$ is minimal over an ideal generated by at most $c-1$ elements. Let $x_2, \dots, x_d$ be lifts of these elts in P.

Then P is minimal over $(x_1, \dots, x_d)$ with $d \leq c$. But $c = \operatorname{codim} P \leq d$ by the Theorem. Thus $d = c$. $\square$

The theorem also leads to an important corollary about UFDs.

Cor: Let R be a (Noetherian) integral domain. R is a UFD iff every codim 1 prime of R is principal.

Pf: ($\Leftarrow$) If every irred. elt is prime, then factorizations are unique (and they exist by Noetherian). Thus, it suffices to show any irreducible $f \in R$ is prime.

Let P be a prime minimal over (f).
Then $\operatorname{codim} P \leq 1$. If $\operatorname{codim} P = 0$, then $P = 0$. Thus $\operatorname{codim} P = 1$, so P is principal: $P = (p)$.

Thus, $f = pu$, some u. Since f is irreducible, u is a unit, so $(f) = (p)$, so f is prime.

Conversely, assume R is a UFD and $P \subseteq R$ a codim 1 prime. Then P is minimal over some (f). Let

$$f = \prod p_i^{e_i}$$

be its prime factorization. Then we showed that the associated primes are (p_i). In particular, the minimal primes over $\operatorname{Ann}_R(R/(f)) = (f)$ are all principal. $\square$